JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 19, No. 2, March–April 1996

Mixed-Strategy Guidance in Future Ship Defense

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Ship defense against highly maneuverable antiship missiles is formulated as an imperfect information zero-sum pursuit–evasion game. The analysis is performed using a planar kinematic model linearized around a nominal collision course. The solution of the game is in mixed strategies. A numerical example, using simplified pure strategy sets, demonstrates that in a one vs one engagement the antiship missile has a nonzero probability of success. The methodology introduced provides new insight required for the development of future antiship missiles and ship defense systems.

Introduction

THE great vulnerability of large ships to a missile attack has been known for several decades. As a consequence, much effort has been invested in the development of active ship defense systems, such as sophisticated gunnery (e.g., Vulcan Phalanx) and guided interceptor missiles, as well as complex countermeasure systems. Though there has not been much operational experience in ship defense, it is estimated that the current active systems would be very efficient against known threats, which all have predictable (almost straight line) trajectories.

In view of state-of-the-art technology, it seems very reasonable that the future generation of antiship missiles will be designed with high maneuverability and they will fly toward their targets, to make their interception difficult, along hardly predictable trajectories. Such an antiship missile, being nearly as maneuverable as a ship defense missile (or even more so), would be able to avoid interception if it had perfect information on the interceptor. Fortunately (from the ship defense point of view), an antiship missile cannot perform an optimal interception avoidance maneuver in the deterministic sense for the following two reasons. The antiship missile, designed to lock-on and home on to hit its designated target, is blind with respect to the ship defense missile. Moreover, since its primary objective is to destroy the target ship, only a part of its full maneuverability can be used for interception avoidance.

The classical approach has been to consider missile guidance and eventual missile avoidance as separated one-sided optimal control problems. For a future ship defense scenario such a decoupling does not seem to be suitable. A differential game formulation that allows a simultaneous analysis of the problem from the view points of both sides can be more useful.

In this paper a ship defense scenario, using guided missiles to intercept antiship threats, is formulated as an imperfect information zero-sum pursuit–evasion game with a trajectory constraint imposed on the blind evader. Since the blind evader (the antiship missile) cannot perform an optimal interception avoidance maneuver, it has to maneuver randomly to make its trajectory unpredictable (within the limits that allow it to hit the target). Such a maneuver, preprogrammed by the designer, can be easily performed as long as the missile is locked on the target ship. Against such a threat the ship defense system must also incorporate random elements in its strategy. For example, if the interception range (from the ship) were set to a fixed value, antiship missiles could be programmed to execute an optimal avoidance maneuver in the deterministic sense and hit the ship successfully. Therefore, the solution of such a game is in mixed strategies that are probability distributions over the sets of respective admissible deterministic (pure) strategies.

Received April 10, 1995; revision received Sept. 18, 1995; accepted for publication Oct. 10, 1995.

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The objective of this paper is to analyze ship defense based on interceptor missiles in a zero-sum pursuit–evasion game formulation. The analysis is carried out using a simplified planar linearized kinematic model with nondimensional variables. The unconstrained version of this model was formulated and solved in the past.[1] That solution, motivated to analyze classical antiaircraft missile engagements (both air-to-air and surface-to-air), served as the basis of several further studies,[2–5] such as extensions to three-dimensional and imperfect information analysis, in the same context.

As the first step, the solution of the perfect information version of the game, the worst case from the defense point of view, is presented. It is followed by the solution of a qualitative imperfect information game yielding mixed strategies and illustrated by a simple numerical example.

Problem Formulation

The ship defense scenario investigated in this paper has three major elements: the target ship T to be defended, the attacking antiship missile A, and the ship defense missile D, launched from the ship. The analysis of this scenario is based on the following set of assumptions:

A-1) The engagement between the two missiles takes place in a horizontal plane (near sea level).

A-2) The antiship missile A is observed by the ship defense system as soon as it starts homing on the target ship T.

A-3) The engagement starts when the ship defense system launches the interceptor missile D against A.

A-4) The velocities of the missiles V_j are constant and their lateral accelerations are limited $|a_j| \le (a_j)_{\max}$, $(j = A, D)$.

A-5) T is either stationary or its velocity is negligible with respect to each of the missile velocities.

A-6) A has instantaneous dynamics, whereas D has first-order dynamics with the time constant τ.

A-7) The trajectory of both missiles can be linearized along the initial line of sight.

A-8) A has perfect information on T but does not have any information on D, whereas D has perfect information on A.

A-9) The interception of A by D must take place between the minimum range $(R)_{\min}$ and the maximum range $(R)_{\max}$ of the ship defense system.

A-10) If the interception fails, A turns to hit and destroy T.

The origin of the coordinate system is located at the target ship T and its x axis is aligned with the initial line of sight. The assumptions A-4 and A-7 imply that the velocity components parallel to the line of sight are almost constant, and as a consequence the final time of the engagement t_f and the interception range x_f are determined by the initial range x_0. Therefore, the equations of motion in the x direction are trivially solved:

$$x_A(t) = x_0 - V_A t \tag{1}$$

$$x_D(t) = V_D t \tag{2}$$

$$\Delta x(t) \triangleq x_A(t) - x_D(t) = x_0 - (V_A + V_D)t \tag{3}$$

$$t_f = \frac{x_0}{V_A + V_D} \tag{4}$$

$$x_f = x_0 - V_A t_f = V_D t_f \tag{5}$$

The equations of motion normal to the initial line of sight and the respective initial conditions are written, based on assumptions A-6 and A-7, as

$$\dot{y}_A = y_1, \qquad y_A(0) = 0 \tag{6}$$

$$\dot{y}_1 = a_A^c, \qquad y_1(0) = V_A\alpha_0 \tag{7}$$

$$\dot{y}_D = y_2, \qquad y_D(0) = 0 \tag{8}$$

$$\dot{y}_2 = y_3, \qquad y_2(0) = V_D\delta_0 \tag{9}$$

$$\dot{y}_3 = \frac{a_D^c - y_3}{\tau}, \qquad y_3(0) = 0 \tag{10}$$

where a_A^c and a_D^c are the commanded lateral accelerations of A and D, respectively,

$$a_A^c = (a_A)_{\max} v, \qquad |v| \le 1 \tag{11}$$

$$a_D^c = (a_D)_{\max} u, \qquad |u| \le 1 \tag{12}$$

The nonzero initial conditions $V_A\alpha_0$ and $V_D\delta_0$ represent the respective initial velocity components not aligned with the line of sight. By assumption A-7 these components are small compared to the components along the line of sight.

The payoff function J of the engagement is the probability that A successfully avoids interception by D. It is defined via a lethality function $\Phi[\cdot]$, depending on the relationship between the lethal radius of the warhead r_l and the miss distance $|y_A(t_f) - y_D(t_f)|$, expressed as

$$\Phi[|y_A(t_f) - y_D(t_f)|] \triangleq \begin{cases} 1 & \text{if} \quad |y_A(t_f) - y_D(t_f)| > r_l \\ 0 & \text{if} \quad |y_A(t_f) - y_D(t_f)| \le r_l \end{cases} \tag{13}$$

$$J = E\{\Phi[|y_A(t_f) - y_D(t_f)|]\} \tag{14}$$

The objective of D is to minimize this payoff, whereas the objective of A is to maximize it, but still be able to hit T. This constraint is expressed by

$$|y_A(t_f) + \dot{y}_A(t_f)t_C| \le \frac{(a_A)_{\max} t_C^2}{2} \tag{15}$$

where t_C is the time needed by A to reach T after a failed interception,

$$t_C \triangleq x_f / V_A = t_f(V_D/V_A) \tag{16}$$

The set of linear differential equations (6–10) can be reduced to a set of only two by using the following nondimensional variables:

$$\nu \triangleq V_D/V_A \tag{17}$$

$$\mu \triangleq \frac{(a_D)_{\max}}{(a_A)_{\max}} \tag{18}$$

$$\Theta = (t_f - t)/\tau, \qquad \Theta(0) = t_f/\tau \triangleq \Theta_0 \tag{19}$$

$$\Theta_C \triangleq t_C/\tau = \nu\Theta_0 \tag{20}$$

$$Z(\Theta) \triangleq \frac{(y_A - y_D) + \tau\Theta(y_1 - y_2) - \tau^2(e^{-\Theta} + \Theta - 1)y_3}{\tau^2 (a_A)_{\max}} \tag{21}$$

$$W(\Theta) \triangleq \frac{y_A + \tau(\Theta + \Theta_C)y_1}{\tau^2(a_A)_{\max}} \tag{22}$$

where $Z(\Theta)$ is the normalized zero-effort miss distance in the D vs A engagement, whereas $W(\Theta)$ has a similar interpretation between A and a static T. Substitution of Eqs. (17–22) into Eqs. (6–10) yields

$$\frac{dZ}{d\Theta} = \mu(e^{-\Theta} + \Theta - 1)u - \Theta v, \qquad Z(\Theta_0) = Z_0 \tag{23}$$

$$\frac{dW}{d\Theta} = -(\Theta + \Theta_C)v, \qquad W(\Theta_0) = W_0 \tag{24}$$

with the following normalized initial conditions:

$$Z_0 = \frac{(V_A\alpha_0 - V_D\delta_0)\Theta_0}{\tau(a_A)_{\max}} \tag{25}$$

$$W_0 = \frac{V_A\alpha_0(\Theta_0 + \Theta_C)}{\tau(a_A)_{\max}} \tag{26}$$

The payoff function (14) and the constraint (15) can be written in the following nondimensional form:

$$J = E\{\Phi[|Z_f|]\} \tag{27}$$

$$|W(\Theta = 0)| \le \Theta_C^2/2 \tag{28}$$

Perfect Information Game

The perfect information version of the just formulated pursuit–evasion game, which represents the worst case from the point of view of the ship defense, has been solved, and the detailed solution is presented in a contemporary paper.[6] In this section the main results, relevant to the solution of the imperfect information game, are summarized.

Recall first the solution of Ref. 1, which has in the nondimensional formulation only a single state variable Z; its dynamics is described by Eq. (23). The solution is governed by a single parameter μ, defined in Eq. (18). The game space is decomposed into two regions $\mathcal{D}_1$ and $\mathcal{D}_0$, its complement. The first region is defined by

$$\mathcal{D}_1 = \{\Theta, Z \mid \Theta \le \Theta_S \cup |Z| \ge Z^*(\Theta)\} \tag{29}$$

where

$$Z^*(\Theta) = \Theta_S + 0.5(\mu - 1)(\Theta^2 - \Theta_S^2) - \mu(e^{-\Theta} + \Theta - 1) \tag{30}$$

and $\Theta = \Theta_S(\mu)$ is the nonvanishing solution of the equation

$$\Theta = \mu(e^{-\Theta} + \Theta - 1) \tag{31}$$

The optimal strategies in $\mathcal{D}_1$ are

$$u^*(\Theta, Z) = v^*(\Theta, Z) = \mathrm{sign}\{Z\}, \qquad Z \ne 0 \tag{32}$$

The segment $0 < \Theta \le \Theta_S$ of the Θ axis ($Z = 0$) is a dispersal line of A (the evader) that may select either a positive or negative maneuver on it. D (the pursuer) must maneuver to the same direction. The optimal outcome, the value of the game, is a unique function of the initial conditions.

In the region $\mathcal{D}_0$ the optimal strategies are arbitrary. All of the trajectories starting in this region go through the same point, $Z = 0$, $\Theta = \Theta_S$, belonging to the dispersal line. Thus, for these trajectories the game has a constant value (function of μ only)

$$J_0^* = \Theta_S - 0.5(\mu - 1)\Theta_S^2 \triangleq M_S(\mu) \tag{33}$$

Based on the observation that in most antiaircraft missile engagements the initial conditions are in D_0, the major conclusion drawn from this analysis has been that if μ is sufficiently large (at least $\mu > 2$), then the guaranteed miss distance, expressed in its nondimensional form by Eq. (33), is negligibly small. Indeed, in most antiaircraft missile designs the missile/target maneuver ratio has been kept on the order of three or higher.

In future antiship defense scenarios (as well as in antiballistic missile defense) such a favorably high maneuver ratio cannot be guaranteed. In effect, a maneuver ratio of the order of unity, or even smaller, is likely to be expected. Therefore, the results of Ref. 1

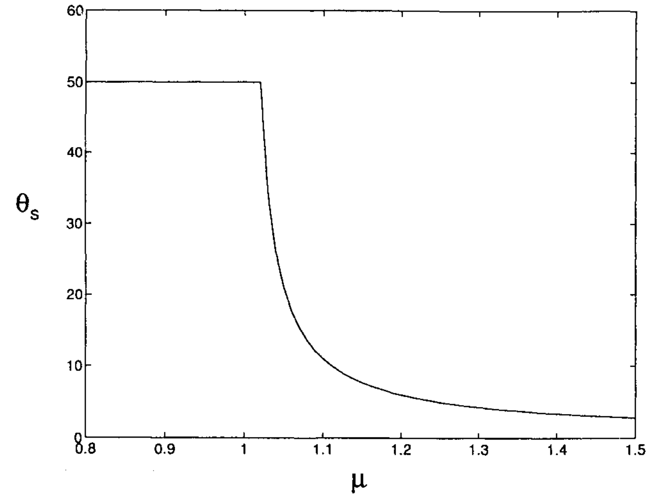

Fig. 1 Results of the unconstrained perfect information game; normalized critical time to go: $\theta_0 = 50$.

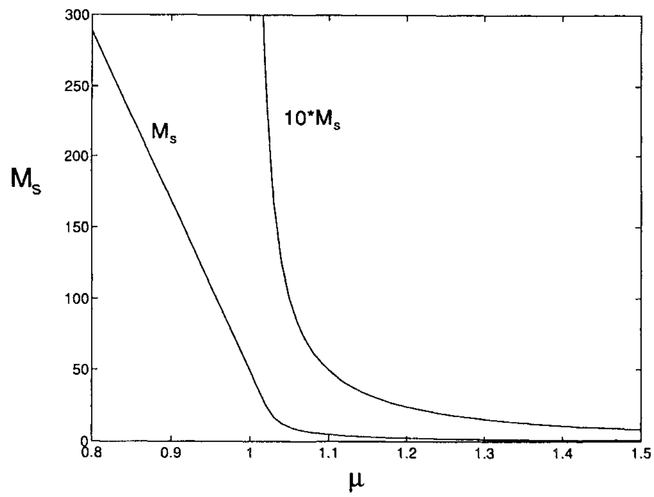

Fig. 2 Results of the unconstrained perfect information game; normalized guaranteed miss distance: $\theta_0 = 50$.

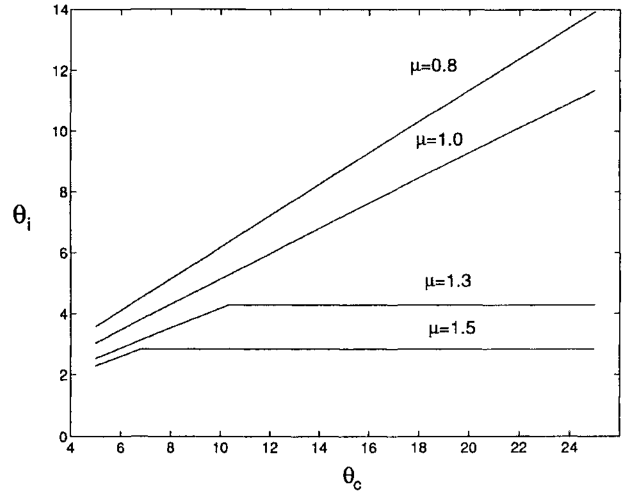

Fig. 3 Results of the perfect information game with constraint; normalized critical time to go.

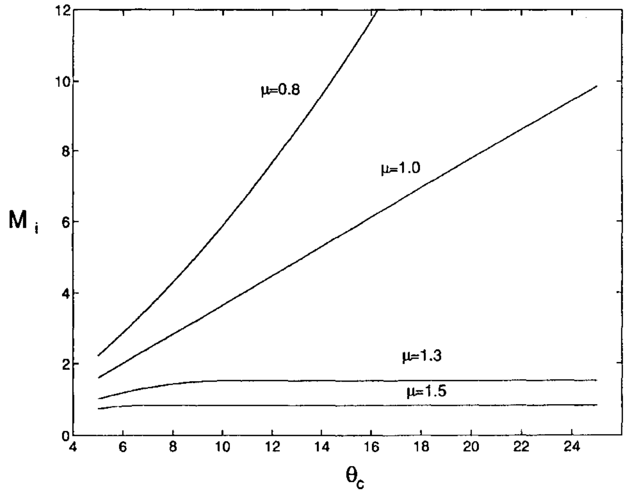

Fig. 4 Results of the perfect information game with constraint; normalized guaranteed miss distance.

have to be reexamined for this range of the parameter μ. In Figs. 1 and 2 the values of Θ_S and M_S are shown for an initial condition of $Z_0 = 0$ and $\Theta_0 = 50$. Note that according to Eq. (31) for $\mu \leq 1$, the value of Θ_S becomes infinite, which means that the entire game space is $\mathcal{D}_1$. As a consequence, Eq. (33) predicting an infinite miss distance is no longer valid. The finite value of the nondimensional miss distance, displayed in Fig. 2, is obtained by integrating Eq. (23) with the appropriate initial conditions. These results show extremely large miss distances, clearly unacceptable for ship defense.

In Ref. 6 the effect of the constraint (15), an inherent feature of the ship defense scenario, was examined in the perfect information game solution. A complete game solution involves the decomposition of the game space into regions of different strategies and finding all of the singular surfaces of the game. The solution depends on two nondimensional parameters, the speed ratio ν and the maneuver ratio μ defined in Eqs. (17) and (18), respectively.

The space of admissible initial conditions is limited by

$$\frac{(R)_{\min}}{\tau V_D} \leq \Theta_0 = \frac{\Theta_C}{\nu} \leq \frac{(R)_{\max}}{\tau V_D} \tag{34}$$

$$|W_0| \leq 0.5(\Theta_0 + \Theta_C)^2 = 0.5(1+\nu)^2\Theta_0^2 \tag{35}$$

There are two cases that have to be distinguished. For the optimal trajectories that never reach the constraint the solution is identical to the unconstrained game solution of Ref. 1. This depends, in general, on the initial condition $W_0 = W(\Theta_0)$ and the parameters of the game, in particular, Θ_C. The constraint becomes active if

$$\Theta_C \leq (\sqrt{2}+1)\Theta_S \tag{36}$$

In this case there are three regions of different strategies in the game. The region where practically all important engagements start has similarities with $\mathcal{D}_0$, i.e., the optimal strategies are arbitrary and the value of the game is constant. All of the trajectories starting in this region may reach the Θ axis at $\Theta = \Theta_i \leq \Theta_S$, the solution of the equation

$$(\sqrt{2}-1)\Theta_C = \mu(e^{-\Theta} + \Theta - 1) \tag{37}$$

A can start to maneuver only at $\Theta = \Theta_i$ to guarantee the normalized miss distance $M_i \leq M_S$. The values of Θ_i and M_i are the functions of Θ_C and μ, as shown in Figs. 3 and 4. This guaranteed miss distance is, of course, smaller then in the unconstrained game, as shown in Fig. 5 for the example of $\mu = 1.3$, $\nu = 1.0$. For a more detailed description of the constrained perfect information game, clearly out of the scope of the present paper, the interested reader is referred to Ref. 6.

It can be seen in Fig. 4 that for small values of the maneuver ratio the guaranteed optimal miss distance is monotonically increasing with Θ_C. The interest of the ship defense system is to keep Θ_C (and, consequently, Θ_0) as small as possible without violating Eq. (34). The best possible solution of the perfect information game for ship defense is the interception of A at $(R)_{\min}$. In such a case A (assumed

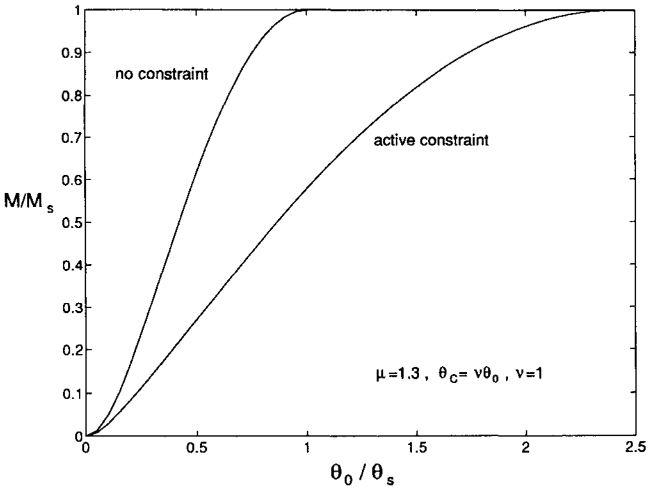

Fig. 5 Effect of the constraint on the guaranteed miss distance.

to have perfect information) will fly along the line of sight toward the ship and will start to maneuver only at $\Theta = \Theta_i$. The maneuver is either to the left or to the right (selected randomly) and guarantees the miss distance predicted for $(R)_{\min}$. Depending on r_l, the lethal radius of the warhead of D, this saddlepoint value of the miss distance may or may not be satisfactory for an effective ship defense. However, since in reality A has no information on D, the actual miss distance will most probably be smaller. This situation is analyzed in the next section.

Imperfect Information Game

In this section it is assumed from the outset that the parameters of the scenario are such that the outcome of the perfect information game is not satisfactory for a reliable ship defense. Otherwise, the analysis of the imperfect information version of the game is trivial. The imperfect information game has an asymmetrical information structure. A can measure only x_A, whereas the ship defense system has access to both x_A and x_D. The fixed parameters of the engagement V_A, V_D, $(a_A)_{\max}$, $(a_D)_{\max}$, $(R)_{\min}$, $(R)_{\max}$, and τ are assumed to be known to all.

The objective of the designer of A is to obtain, in spite of the lack of information, a high probability of interception avoidance. This can be achieved by preprogramming an evasive maneuver strategy $v(x_A)$ that generates a trajectory $y_A(x_A)$ that will not violate the constraint

$$|y_A(x_A) + (x_A/V_A)\dot{y}(x_A)| \leq \tfrac{1}{2}(x_A/V_A)^2(a_A)_{\max} \tag{38}$$

which is an equivalent of Eq. (15).

A simplified discretized model of the imperfect information pursuit–evasion game was formulated and analyzed in Ref. 7. It shows that even for a large maneuver advantage of A ($\mu = 0.5$) the probability of a successful interception can be quite substantial.

The results of the perfect information game[6] indicate that, to be able to obtain a reasonably large miss distance (of the order of M_S, or M_i), A must maneuver before the interception at least for a duration of Θ_S (or Θ_i) to the same direction. Maneuvering near the constraint, as well as any nonmaneuvering period, makes the trajectory of A predictable, resulting in a very small or even zero miss distance. Nevertheless, a very long maneuver to the same direction may allow to the ship defense system to estimate the future position of A with good accuracy and may lead to a similar result. Moreover, any change in the direction of the maneuver, not sufficiently far away (determined by Θ_S) from the unknown point of the interception presents a risk of generating a reduced (or even zero) miss distance.

These observations serve as guidelines to the antiship missile designer, who may wish to adopt the following structure for the evasive strategy of A:

1) The optimal maneuvering sequence will include a small number of randomly timed direction changes (switches).

2) The sequence of alternating maximal maneuvers starts at the distance

$$(x_A)_{\max} \triangleq (R)_{\max} + \tau\Theta_S V_A \tag{39}$$

from the target ship T. At that time A should be aligned with the line of sight to T. [There is no advantage to start the maneuver earlier, and a later start may allow a successful interception at $(R)_{\max}$.]

3) Each single maneuver will have a duration of the order of Θ_S (or Θ_i).

4) No single maneuver will be longer than the time to go of an interception by a simultaneously launched ship defense missile.

5) The trajectory generated by the evasive maneuvering sequence will not reach the constraint (38) before reaching a critical distance x_A^* from T,

$$x_A^* = (R)_{\min}(1 + 1/\nu) \tag{40}$$

Conditions 2–5 represent different inequalities that may not always be compatible with each other. Their objective is to minimize the risk of A being intercepted, as well as not to rule out the possibility of reaching the guaranteed miss distance of the perfect information game. Even if all of them can be simultaneously satisfied, this is by no means a sufficient condition for an efficient and successful antiship strategy. This goal can be achieved only by the randomness of the maneuvers as pointed out in condition 1. If some of the conditions 2–5 cannot be satisfied simultaneously, the designer of A must search for an optimal compromise.

Thus, it can be summarized that the parameters of the strategy of A are randomly selected distances $(x_A)_i (i = 1, 2, \ldots, k)$, for changing the direction of the evasive maneuver. It can be assumed, without loss of generality, that the first maneuver starts according to Eq. (39).

The ship defense system is assumed to have perfect information on the state of the antiship threat. The radar system of the ship measures the position and the velocity vector of A with high accuracy. Moreover, if A performs long maneuvers, the effect of a continuous lateral acceleration can be accounted for in predicting the point of a future interception. Based on this information, as well as taking into account the trajectory constraint imposed on A, the ship defense system has to decide when and in which initial direction D will be launched.

In a perfect information scenario the optimal launch direction is toward the predicted collision point, computed for the current velocity vector of A (without considering any future maneuver, but taking into account the constraint), i.e., with Z_0 on the dispersal surface.

In the imperfect information scenario of interest (namely, an unsatisfactory deterministic outcome and randomly maneuvering A), a nonzero initial condition bias δ_0, based on a presumed continuous maneuver, may be considered. The magnitude of this bias and its sign will be random variables.

Obviously (see Introduction), the time for launching D has to be selected randomly to satisfy

$$(R)_{\min} \leq \Theta_0\tau V_D \leq (R)_{\max} \tag{41}$$

The guidance law of D, once it is launched, either can be a deterministic one or can incorporate a random bias consistent with the bias in the initial condition. The deterministic part of the guidance law should be the one obtained from the solution of the perfect information game. In the region of arbitrary pursuer strategies this guidance law can be specified as a linear one with time-dependent gain, as proposed in Refs. 1 and 5.

If, for any reason, a bias in the launch direction (or in the guidance law) of D is not considered as a viable option, the imperfect information game of interest becomes a game of timing.[8–9] The simplest version of such a game is obtained if one assumes a scenario where A has only one randomly selected maneuver switch. In this case the relevant game space can be transformed to the unit square. Any point in this square corresponds to a pair of randomly selected events (the maneuver switch of A and the launching of D), leading to a well-defined deterministic outcome ($\Phi = 0$ or $\Phi = 1$), depending on the relationship between the lethal radius of the warhead r_l and the miss distance. The solution of this game of timing consists of the pair of the optimal probability distributions corresponding, respectively, to each of the two events and the resulting saddle value of the stochastic payoff function (14). This is illustrated in the following example, using much simplified pure strategy sets.

Numerical Example

The parameters used in this illustrative example are as follows: $\tau = 0.5$ s, $V_A = 1200$ m/s, $V_D = 800$ m/s, $(a_A)_{\max} = 300$ m/s^2, $(a_D)_{\max} = 390$ m/s^2, $(R)_{\max} = 8000$ m, and $(R)_{\min} = 2000$ m.

Based on these data the nondimensional parameters in the perfect information game solution are $\Theta_S = 4.27$ and $M_S = 1.53$ (for the unconstrained game version), $(\Theta_C)_{\min} = 3.33$, $\Theta_i = 1.91$, and $M_i = 0.69$ (for the game with constraint). The dimensional values of the guaranteed miss distances for the two deterministic game versions are 115 m (unconstrained) and 52 m (constrained), respectively, both equally inadmissible for ship defense.

Taking into account these results, a very simple example of the imperfect information game is solved. For sake of simplicity it is assumed that A is programmed to execute only a single maneuver [as long as the constraint (38) is not reached] starting at a randomly selected distance $(x_A)_m$ from the ship in the domain,

$$(x_A)_{\min} \leq (x_A)_m \leq (x_A)_{\max} \tag{42}$$

with $(x_A)_{\min} = (R)_{\min} = 2000$ m, and based on Eq. (39), $(x_A)_{\max} =$ 10,500 m.

It means that A has three trajectory phases: a straight line until $(x_A)_m$ is reached, followed by a maneuver to an arbitrarily selected direction (left or right) until the constraint (38) is reached, and a final maneuvering phase to the opposite direction satisfying the constraint.

Facing this situation the strategy of the ship defense is to select randomly the time when the ship defense missile D is launched. This time determines the final distance x_f from the ship, where the interception takes place, satisfying

$$(R)_{\min} \leq x_f \leq (R)_{\max} \tag{43}$$

which also defines the time of the interception. Moreover, since the ship defense system has perfect information on A when D is launched, it selects the launch direction in the following way:

1) If A has not yet started to maneuver, D is launched toward a collision course (with Z_0 on the dispersal surface).

2) If A is already maneuvering, the direction of the launch is toward the predicted point of impact, taking into account continued maneuvering of A until the constraint (38) is reached.

Note that because of the simplifying assumptions just outlined, the pure strategy sets of both players become restricted. For illustrative purposes, however, it seems to be useful because it captures the major salient elements of the scenario.

The decision space associated with this game (measured in kilometers) is the quadrangle of $2.0 \leq (x_A)_m \leq 11.0, 2.0 \leq x_f \leq 8.0$. Based on the results computed for each eventual engagement, using the linearized kinematic model of Eqs. (1–12), this quadrangle can be divided into three regions, as seen in Fig. 6. In two of the regions the resulting miss distance is identically zero. The upper left-hand part is characterized by trajectories where A starts to maneuver before D is launched, whereas the lower right-hand part is characterized by A being hit before starting to maneuver. Between these regions the miss distance depends on the actual time elapsed between the beginning of the evasive maneuver of A and the interception that can reach very large values, as indicated by the results of the perfect information game.

By assuming that the lethal radius of the warhead of D is $r_l = 10$ m, the decision space of the game can be divided to winning zones of A and D, respectively. In Fig. 7 the boundaries of the winning zone of A (the lines of miss distance = 10 m) are plotted, comparing simulation results obtained from the nonlinear kinematical equations with the results computed for the linearized kinematic model. The comparison indicates the usefulness of the simplified mathematical model of Eqs. (1–12), based on linearized kinematics.

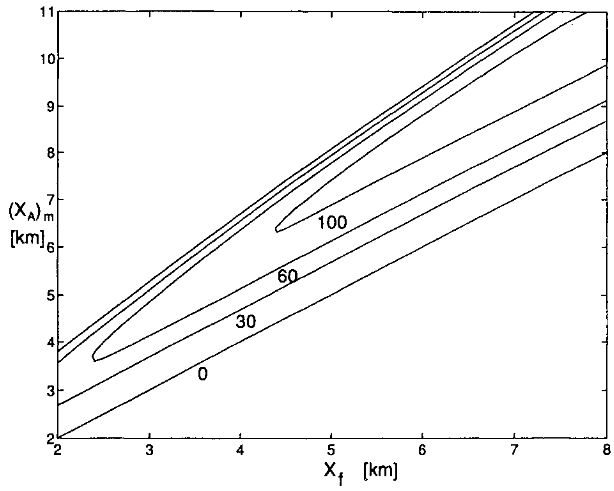

Fig. 6 Constant miss distance lines in the decision space of the imperfect information game (linearized model).

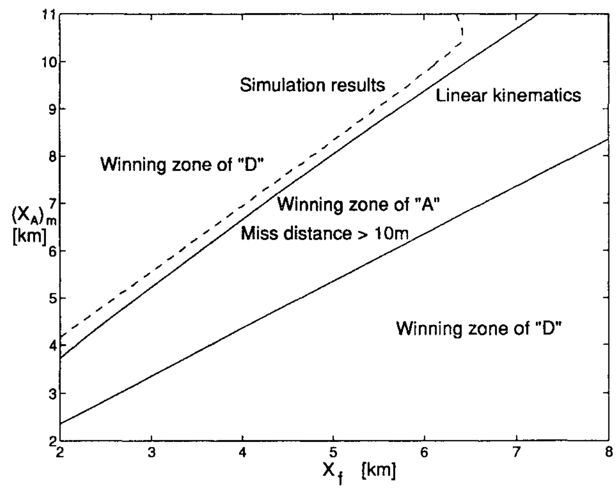

Fig. 7 Winning zones in the decision space of the imperfect information game.

The solution of the qualitative game is rather straightforward. The designer of A can determine a finite number of (say, n) values for $(x_A)_m$, such that at least one of them always provides a satisfactory outcome against any value of x_f that can be chosen by the ship defense system. The optimal solution consists of finding the smallest number for n based on a simple geometrical construction. This optimal value of n is uniquely determined by the parameters of the problem. The solution [i.e., the actual $(x_A)_m$ values, elements of the optimal segments, the optimal pure strategy set of A] is, however, generally not unique, as it is shown in the sequel. The optimal mixed strategy for a given n is to select one of these values randomly, with equal probabilities. Thus, the probability that A will avoid a single ship defense interceptor (and later hit the ship) is at least $1/n$.

The optimal mixed strategy of the ship defense system is also simple. In the domain of x_f, defined by Eq. (43), n disjoint segments can be identified in a way that no value of $(x_A)_m$ can yield a victory for A against more than one segment. These segments represent the optimal pure strategy set of D. If the ship defense system selects the time for launching D with an equal probability $(1/n)$ from each such segment, then it is guaranteed that the probability of success for A will not be greater than $(1/n)$, which is the saddle-point value of the imperfect information game.

For the present numerical example the optimal pure strategy sets of A and D are found, both for the linearized kinematic model and for the simulation results obtained from the nonlinear kinematical equations. The respective solutions (obtained by using a simple geometrical construction, illustrated by the dashed lines) are displayed in Figs. 8 and 9.

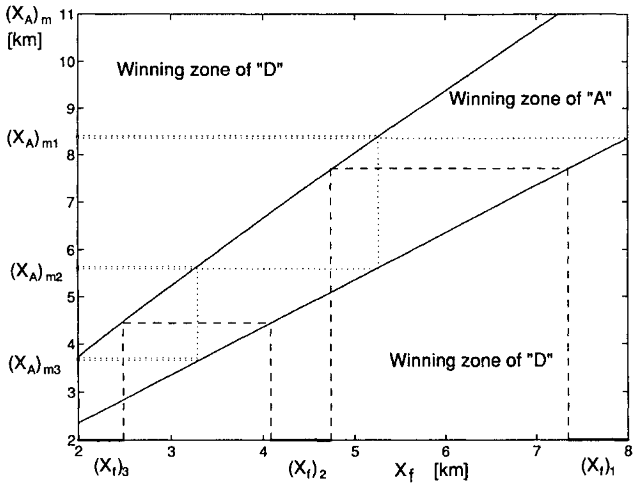

Fig. 8 Example of optimal mixed strategy solution in the imperfect information game, linearized model.

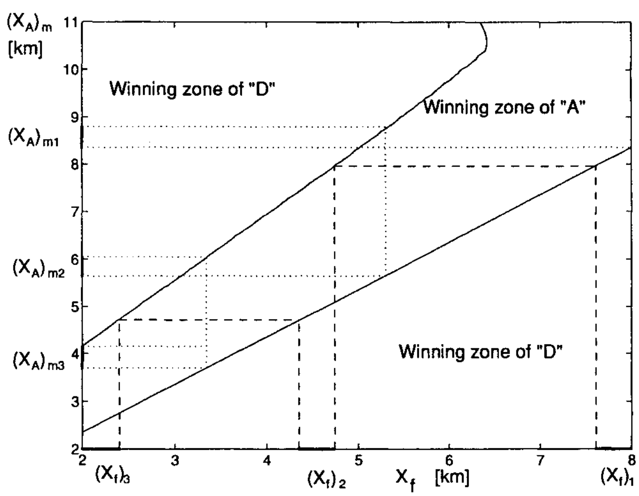

Fig. 9 Example of optimal mixed strategy solution in the imperfect information game, simulation results.

For both cases $n = 3$ and the optimal segments are, though not identical, quite similar (here we chose equal segments in each case, although it is not necessary). The exact numerical results (in kilometers) for the linearized model are

$$8.36 \leq (x_A)_{m1} \leq 8.41, \qquad 5.60 \leq (x_A)_{m2} \leq 5.65$$

$$3.65 \leq (x_A)_{m3} \leq 3.68, \qquad 7.35 \leq x_{f1} \leq 8.0$$

$$4.08 \leq x_{f2} \leq 4.74, \qquad 2.0 \leq x_{f3} \leq 2.49$$

and for the nonlinear simulation

$$8.36 \leq (x_A)_{m1} \leq 8.8, \qquad 5.65 \leq (x_A)_{m2} \leq 6.05$$

$$3.69 \leq (x_A)_{m3} \leq 4.15, \qquad 7.61 \leq x_{f1} \leq 8.0$$

$$4.35 \leq x_{f2} \leq 4.74, \qquad 2.0 \leq x_{f3} \leq 2.40$$

The optimal mixed strategy for both players is to select one of the optimal segments with probability $\frac{1}{3}$ and then chose an arbitrary element from this segment. The value of the game (the probability of A escaping interception from single interceptor) is $\frac{1}{3}$.

Conclusions

This paper analyzes a future ship defense scenario using guided interceptor missiles against highly maneuverable antiship missiles. The engagement is formulated as a zero-sum pursuit–evasion game with a state constraint on the evader (the antiship missile).

In the perfect information version of the game the interception of the antiship missile cannot be guaranteed. For the realistic, imperfect information formulation of the scenario, the optimal strategy of the antiship missile consists of a small number of randomly selected hard maneuvers respecting the constraint. Against such a random strategy the ship defense system must also randomly select the time (and sometimes the direction also) of launching the interceptor missiles against the threat. Including random elements in the guidance law of the interceptor can also be considered.

By judicious selection of the available parameters the designer of the antiship missile can determine an optimal pure strategy set and an optimal mixed strategy guaranteeing a nonvanishing probability of success in a one vs one engagement. This is demonstrated by the results of an illustrative example with simplified sets of pure strategies.

To achieve a reliable ship defense more than one interceptor has to be launched nonsimultaneously against each antiship threat. A randomized shoot-look-shoot strategy (which is also a mixed strategy in the mathematical sense) can minimize the average number of interceptors needed against a given number of threats.

The ideas and the methodology presented in this paper create a fresh insight that paves the way toward the development of improved guidance strategies both for antiship missiles and ship defense interceptors.

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